$$\int_0^{\tau_i} \lambda_i(t, \tau_i, s, v(s), z_i^*, m_i^3)\, ds < 1, \quad \forall \tau_i \in [0, \theta], \quad i = 1, 2, 3$$

Therefore the capture of the pursued is possible in the present game only when the instant of switchover $\tau_i \in (0, \theta]$, e.g. $\tau_i = \tau\ (i = 1, 2, 3)$.

REFERENCES

1. NIKOL'SKII M.S., Linear differential games with variable structure. Dokl. Akad. Nauk SSSR, 276, 4, 1984.
2. SUBBOTIN A.I. and CHENTSOV A.G., Optimal Guarantees in Control Problems. Moscow, Nauka, 1981
3. GRIGORENKO N.D., Differential Games of Pursuit by Several Objects. Moscow, Izd-vo MGU, 1983.

Translated by L.K.

PMM U.S.S.R., Vol.50, No.1, pp.115-118, 1986
Printed in Great Britain

0021-8928/86 $10.00+0.00

ON CONSTRUCTING A FUNCTIONAL IN THE PROBLEM OF OPTIMAL CONTROL*

ZH. KH. BAIMURZAYEVA

The problem of constructing a functional in the theory of control of material systems is considered as an inverse problem of dynamics /1/. It was A.M. Letov /2/ who first became aware of the practical value of the inverse problems in optimal control. He solved a number of inverse problems of choosing the optimal functional in problems of controlling aircraft. The approach was also successfully used in problems of robotics /3/. The procedure used in solving inverse problems makes it possible to combine the merits and virtues of the engineering problems based on formulating the control laws from the conditions of motion according to a given program, with the possibilities offered by methods of optimal control theory.

Let us consider a controlled object described by a system of ordinary differential equations

$$x^{\cdot} = f(x, u, t), \quad x \in R^n, \quad u \in R^r \tag{1}$$

where f is a continuously differentiable vector function, u is a piecewise continuous control and $0 \leqslant t \leqslant T$. The initial condition $a = x(0)$ and time T will be assumed given. We shall call the pair $\{u(t), x(t)\}$ the admissible process, if $u(t), x(t)$ satisfy (1).

We shall treat the inverse problem of optimal control for the object in question as a problem of determining a continuously differentiable function $f_0(x, u)$, such that the solution of the problem of maximizing the functional

$$J = \int_0^T f_0(x, u)\, dt \tag{2}$$

leads to the given admissible process $\{u^*(t), x^*(t)\}$.

Let us write the unknown function $f_0(x, u)$ as the sum of the continuously differentiable functions $\varphi_s(x, u)\ (s = 1, 2, \ldots, p)$ in the form

$$f_0(x, u) = \sum_{s=1}^{p} c_s \varphi_s(x, u)$$

In this case the inverse problem of optimal control will be reduced to finding all coefficients $c_s\ (s = 1, 2, \ldots, p)$, such that the maximization of the functional

$$J = \int_0^T \sum_{s=1}^{p} c_s \varphi_s(x, u)\, dt \tag{3}$$

when (1) is satisfied, leads to the given admissible process $\{u^*(t), x^*(t)\}$, $0 \leqslant t \leqslant T$.

In order to exclude the trivial case where all coefficients c_s are zero (in this case any admissible process will be optimal for a functional identically equal to zero), we shall introduce the concept of a non-degenerate solution of the inverse problem: we will call the solution of the inverse problem non-degenerate, if at least one of the coefficients c_s is not

*Prikl.Matem.Mekhan., 50, 1, 159-163, 1986 zero.

Let us assume that the coefficients solving the inverse problem have been found. Then the conditions of the L.S. Pontryagin maximum principle /4/ will represent the necessary conditions of optimality of the given admissible process $\{u^*(t), x^*(t)\}$, $0 \leqslant t \leqslant T$ functional (3) when (1) is satisfied. These conditions are: if $\{u^*(t), x^*(t)\}$ is an optimal process, then there exists a non-zero solution of the system of conjugate equations

$$\psi_l^{\cdot}(t) = -\sum_{\nu=1}^{n} \psi_\nu(t) \frac{\partial f_\nu(*)}{\partial x_l} - \sum_{s=1}^{p} c_s \frac{\partial \varphi_s(*)}{\partial x_l} \tag{4}$$

with the boundary conditions $\psi_l(T) = 0$ $(l = 1, 2, \ldots, n)$, such, that

$$u^*(t) = \operatorname{Arg} \max_{u \in R^r} \Big[\sum_{s=1}^{p} c_s \varphi_s(x^*, u) + \sum_{\nu=1}^{n} \psi_l(t) f_\nu(x^*, u, t) \Big]$$

In the present case (the set of admissible controls is open) from (4) it follows that

$$\sum_{s=0}^{p} c_s \frac{\partial \varphi_s(*)}{\partial u_j} + \sum_{\nu=1}^{n} \psi_\nu(t) \frac{\partial f_\nu(*)}{\partial u_j} = 0 \quad (j = 1, 2, \ldots, r) \tag{5}$$

where the asterisk indicates that the values of the arguments of the derivatives are chosen on the trajectory $\{u^*(t), x^*(t)\}$.

Let us consider the homogeneous system of equations

$$\psi_l^{\cdot}(t) = -\sum_{\nu=1}^{n} \psi_\nu(t) \frac{\partial f_\nu(*)}{\partial x_l} \quad (l = 1, 2, \ldots, n) \tag{6}$$

We denote the fundamental matrix of solutions of (6), which becomes a unit matrix at a given value of transition time $t = T$, by $\Psi(t)$. Then the solution of system (4) with prescribed initial condition can be written in the form $\Psi(t) = \Psi_c(t)\, c$ where $\Psi(t) = (\psi_1(t), \psi_2(t), \ldots, \psi_r(t))^T$ is the column vector of conjugate variables, $c = (c_1, c_2, \ldots, c_p)^T$ is a column vector with the unknown coefficients of the functional (3) serving as the coordinates, and the matrix $\Psi_c(t)$ has the form

$$\Psi_c(t) = \Psi(t) \int_0^T \Psi^{-1}(t)\, \Phi_x(t)\, dt$$

with the matrix $\Phi_x(t)$ appearing in it given by

$$\Phi_x(t) = \left[\frac{\partial \varphi_s(*)}{\partial x_l} \right] \quad (l = 1, 2, \ldots, n;\ s = 1, 2, \ldots, p) \tag{7}$$

We shall also introduce the matrices defined over the given process $\{u^*(t), x^*(t)\}$, in the form

$$\Phi_u(t) = \left[\frac{\partial \varphi_s(*)}{\partial u_j} \right], \quad F_u(t) = \left[\frac{\partial f_\nu(*)}{\partial u_j} \right] \quad (j = 1, 2, \ldots, r;$$
$$\nu = 1, 2, \ldots, n).$$

Eq.(5), in the above notation, takes the form

$$\Phi_u(t)\, c + F_u(t)\, \Psi(t) = 0$$

Substituting the solution for $\Psi(t)$ found into the relation obtained, we arrive at the equation

$$K(t)\, c = 0, \quad c = (c_1, c_2, \ldots, c_p)^T, \quad K(t) = [k_{js}] = \Phi_u(t) + F_u(t)\, \Psi_c(t) \tag{8}$$

We shall formulate the result obtained in the form of a theorem, assuming that the conditions imposed on the function f, φ_s, given in the beginning, are satisfied.

Theorem 1. Any solution of the inverse problem of optimal control (1), (3) will be, at $0 \leqslant t \leqslant T$, a solution of the system of linear homogeneous Eqs.(8). If the functions $k_{j1}(t)$, $k_{j2}(t), \ldots, k_{jp}(t)$ are linearly independent, for some $1 \leqslant j \leqslant r$, then the inverse problem has no degenerate solutions.

To determine the possible values of the coefficients satisfying system (8), we will consider the i-th equation of this system, which has the form

$$k_{i1}(t)\, c_1 + k_{i2}(t)\, c_2 + \ldots + k_{ip}(t)\, c_p = 0 \quad (i = 1, 2, \ldots, r). \tag{9}$$

Introducing the scalar product

$$(x(t), y(t)) = \int_0^T x(t)\, y(t)\, dt$$

we multiply Eqs.(9), one after the other, by $k_{i1}(t), k_{i2}(t), \ldots, k_{ip}(t)$. This yields a system of linear homogeneous equations. Let us denote the matrices of these systems (Gram matrices of the system of functions $k_{i1}(t), k_{i2}(t), \ldots, k_{ip}(t)$ in the interval $(0, T)$) by Γ_i, and write them in the form

$$\Gamma_i c = 0 \tag{10}$$

Let us consider the first system of (10). Let $\rho_1 = \operatorname{rank} \Gamma_1$. The general solution of system (10) is obtained in the form $c = R_1 w_1$, where $w_1 \in R^{p-\rho_1}$ and the columns of the matrix R_1 represent the fundamental system of solutions of (10). If the rank of ρ_1 is p, then the system in question has no non-zero solutions and the inverse problem has no degenerate solutions.

Let us suppose that a vector c is found, satisfying $j\,(1 \leqslant j \leqslant r)$ systems of homogeneous Eqs.(10), i.e. that a matrix R_j is known in the representation of the solution

$$c = R_j w_j \tag{11}$$

for an arbitrary vector w_j. Substituting (11) into the $(j+1)$-th equation of (10), we obtain a linear homogeneous system for determining the vector w_j such that the vector c will be a solution of not only the j-th system, but also of the $(j+1)$-th system of (10) of the form

$$\Gamma_{j+1} R_j w_j = 0 \tag{12}$$

Having found the general solution of (12) in the form $w_j = W_j w_{j+1}$ (columns of the matrix W_j represent the fundamental system of solutions and w_{j+1} is an arbitrary vector), we obtain the solution for the $(j+1)$-th system of (10) in the form

$$c = R_{j+1} w_{j+1},\ R_{j+1} = R_j W_j$$

Writing, consecutively, $j = 1, 2, \ldots, r$, we finally obtain $c = R_r w_r$. The formula obtained solves the problem of determining all vectors c satisfying the necessary conditions of optimality. If we find for some $j = 1, 2, \ldots, r-1$ that system (12) has no solution apart from the trivial one, then the inverse problem has no degenerate solutions.

Let us consider the same controlled obuect described by the system of Eqs.(1) with given initial condition $a = x(0)$ and known transition time T. We shall now assume that the following constraints are imposed on the controls $u_j(t)\ (j = 1, 2, \ldots, r)$:

$$\alpha_j \leqslant u_j(t) \leqslant \beta_j,\ 0 \leqslant t \leqslant T \tag{13}$$

We define the inverse problem of optimal control for such an object as the problem of determining the coefficients $c_s\ (s = 1, 2, \ldots, p)$ in the functional (3) such that the maximization of this functional, when (1), (13) are satisfied, leads to the given process $\{u^*(t), x^*(t)\}$. In the present case condition (5) is no longer than necessary condition for the maximum of the Hamilton function on the optimal control, and in order to formulate such a condition we shall find, using the given admissible process and the constraints (13), the sets

$$T_j^\alpha = \{t \mid u_j(t) = \alpha_j\},\ \ T_j^\beta = \{t \mid u_j(t) = \beta_j\},\ \ T_j = \{t \mid \alpha_j < u_j(t) < \beta_j\}$$
$$t \in [0, T],\ j = 1, 2, \ldots, r$$

We shall assume that for every $j = 1, 2, \ldots, r$ the sets $T_j^\alpha,\ T_j^\beta,\ T_j$ introduced above are either empty, or that they can be written in the form of a union of a finite number of pair-wise intersecting intervals $T_{jk}^\alpha,\ T_{jk}^\beta,\ T_{jk}$, i.e. the following expansion holds:

$$T_j^\alpha = \bigcup_{k=1}^{n_i^\alpha} T_{jk}^\alpha,\quad T_j^\beta = \bigcup_{k=1}^{n_i^\beta} T_{jk}^\beta,\quad T_j = \bigcup_{k=1}^{n_i} T_{jk}$$
$$T_{jk}^\alpha = \{t \mid t_{jk}^{(\alpha_1)} \leqslant t \leqslant t_{jk}^{(\alpha_2)}\},\quad T_{jk}^\beta = \{t \mid t_{jk}^{(\beta_1)} \leqslant t \leqslant t_{jk}^{(\beta_2)}\}$$
$$T_{jk} = \{t \mid t_{jk}^{(1)} \leqslant t \leqslant t_{jk}^{(2)}\}$$

Using the above assumptions we shall formulate the necessary conditions for the coefficients $c_1, c_2, \ldots, c_p$ to solve the given inverse problem of optimal control.

Theorem 2. Let the coefficients $c_1, c_2, \ldots, c_p$ be a solution of the inverse problem of optimal control with constraints (13), for the given admissible process $\{u^*(t), x^*(t)\}$. Then the coefficients will satisfy the relations

$$\Sigma = 0,\ t \in T_j;\ \Sigma \leqslant 0,\ t \in T_j^\alpha \tag{14}$$
$$\Sigma \geqslant 0,\ t \in T_j^\beta;\ \Sigma = \sum_{s=1}^{p} k_{js}(t)\, c_s$$

in which the coefficients $k_{js}(t)$ have the form

$$k_{js}(t) = \frac{\partial \varphi_s(*)}{\partial u_j} + \sum_{\nu=1}^{n} \psi_{\nu s}^c(t) \left[\frac{\partial f_\nu(*)}{\partial u_j}\right] \tag{15}$$

The elements $\psi_{\nu s}{}^c(t)$ of the matrix $\Psi_c(t)$ are found from its expression

$$\Psi_c(t) = \Psi(t) \int_0^T \Psi^{-1}(t)\, \Phi_x(t)\, dt \tag{16}$$

where $\Psi(t)$ is the fundamental matrix of solutions of the system of homogeneous differential Eqs.(6), becoming a unit matrix when $t = T$, and $\Phi_x(t)$ is the matrix (7).

Proof. The necessary conditions of optimality of the process $\{u^*(t), x^*(t)\}$ for the functional (3), when conditions (13) are satisfied, can be formulated as follows: if the admissible process $\{u^*(t), x^*(t)\}$ is optimal, then there exists a non-zero solution of the system of conjugate equations

$$\psi_l^{\cdot}(t) = -\sum_{\nu=1}^{n} \psi_\nu(t) \left[\frac{\partial f_\nu(*)}{\partial x_l}\right] - \sum_{s=1}^{p} c_s \left[\frac{\partial \varphi_s(*)}{\partial x_l}\right] \tag{17}$$
$$\psi_l(T) = 0, \ l = 1, 2, \ldots, n$$

such, that for all, apart from a finite number of values of $t\,(0 \leqslant t < T)$, the following relations hold:

$$E = 0, \ t \in T_j; \ E \leqslant 0, \ t \in T_j^{\alpha}; \ E \geqslant 0, \ t \in T_j^{\beta} \tag{18}$$
$$E = \sum_{s=1}^{p} c_s \frac{\partial \varphi_s(*)}{\partial u_j} + \sum_{\nu=1}^{n} \psi_\nu(t) \frac{\partial f_\nu(*)}{\partial u_j}$$

Relations (18) represent the Kuhn-Tucker conditions for the Hamilton function to attain its maximum on the optimal control, and can be obtained from theorem 4.14 /5/ for the type of constraints imposed on the controls considered here.

The solution of the system of differential Eqs.(17) with the boundary conditions shown, has the form

$$\Psi(t) = \psi_c(t)\, c \tag{19}$$

where the notation of (16) and (7) is used.

The matrix $\Psi(t)$ appearing here is the fundamental matrix of solutions of system (17), becoming a unit matrix when $t = T$, $\Psi(t) = (\psi_1(t), \ \psi_2(t), \ldots, \psi_n(t))^T$, $c = (c_1, c_2, \ldots, c_p)^T$.

Subsituting (19) into (18) and using the notation (15), we arrive at the statement of the theorem.

Corollary. Suppose, for some $j \in \{1, 2, \ldots, r\}$ and $k \in (1, 2, \ldots, n_j)$, the system of functions $k_{j1}(t), k_{j2}(t), \ldots, k_{jp}(t)$ is linearly independent in the interval T_{jk}. Then the inverse problem has no non-degenerate solutions.

Note. In some cases the inverse problem is formulated as the problem of finding the non-negative coefficients $c_s \geqslant 0$ in the functional (3). clearly, in this case the necessary condition of optimality of existence of the non-degenerate solution formulated in the statement of the theorem will also hold.

REFERENCES

1. GALIULLIN A.S., Inverse Problems of Dynamics. Moscow, Nauka, 1981.
2. LETOV A.M., Dynamics of Flight and Control. Moscow, Nauka, 1969.
3. BELETSKII V.V., Bipedal Walk: Model Problesm of Dynamics and Control. Moscow, Nauka, 1984.
4. PONTRYAGIN L.S., BOLTYANSKII V.G., GAMKRELIDZE R.V. and MISHCHENKO E.D., Mathematical Theory of Optimal Processes. Moscow, Fizmatgiz, 1961.
5. BOLTYANSKII V.G., Mathematical Methods of Optimal Control. Moscow, Nauka, 1969.

Translated by L.K.